Hamiltonian Formulation and Statistics of an Attracting System of Nonlinear Oscillators

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An attracting system of r nonlinear oscillators of an extended van der Pol type is investigated with respect to Hamiltonian formulation. The case of $r = 2$ is rather simple, though nontrivial. For $r > 2$ the tests with Jacobi's identity and Frechet derivatives are negative if Hamiltonians in the natural variables are looked for. Independently, a Liouville theorem is proved and equilibrium statistics is made possible, which leads to a Gaussian distribution in the natural variables.

I. Introduction

A class of systems of r nonlinear oscillators of an extended van der Pol type has already been introduced by the author [1]. The interaction between the oscillators was given by matrices containing cubic nonlinearities. Investigation of the Lyapunov stability resulted, under certain conditions, in defining an attracting system [2, 3] in which the driving and damping of the van der Pol oscillators exactly cancel out. The attracting system is of the form

$$\ddot{Y} + [(Y, Y)\, M_a + (\dot{Y}, \dot{Y})\, N_a - P_a]\, \dot{Y} + Y = 0\,, \tag{1}$$

where Y is a real vector of arbitrary length r and M_a, N_a and P_a are antisymmetric $r \times r$ real matrices. In [2] it was shown that system (1) is completely integrable for $r = 2$, and in [3] strong arguments for nonintegrability were given for $r > 2$.

This contribution is essentially devoted to the Hamiltonian and variational formulations of system (1) for $r \geqq 2$. The $r = 2$ case is given in Section II and a discussion of the $r > 2$ case is the topic of Section III. Section IV is on the statistics of system (1).

II. Hamiltonian Formulation for $r = 2$

If M_a, P_a, and N_a are defined more explicitly as

$$M_a = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \quad N_a = \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}, \quad P_a = \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}, \tag{2}$$

then system (1) becomes

$$\ddot{y}_1 + [m(y_1^2 + y_2^2) + n(\dot{y}_1^2 + \dot{y}_2^2) + p]\, \dot{y}_2 + y_1 = 0\,, \tag{3}$$

$$\ddot{y}_2 - [m(y_1^2 + y_2^2) + n(\dot{y}_1^2 + \dot{y}_2^2) + p]\, \dot{y}_1 + y_2 = 0\,. \tag{4}$$

Let us first check whether the Frechet [4] derivative F of system (3, 4) is a symmetric operator. Perturbing $y_1 \to y_1 + u$ and $y_2 \to y_2 + v$, we have

$$F = \left[\begin{pmatrix} \dfrac{\partial^2}{\partial t^2} & 0 \\ 0 & \dfrac{\partial^2}{\partial t^2} \end{pmatrix} + \begin{pmatrix} 2n\, \dot{y}_1 \dot{y}_2 \dfrac{\partial}{\partial t} & [m(y_1^2 + y_2^2) + n(\dot{y}_1^2 + 3\dot{y}_2^2) + p] \dfrac{\partial}{\partial t} \\ -[m(y_1^2 + y_2^2) + n(3\dot{y}_1^2 + \dot{y}_2^2) + p] \dfrac{\partial}{\partial t} & -2n\, \dot{y}_1 \dot{y}_2 \dfrac{\partial}{\partial t} \end{pmatrix} \right.$$
$$\left. + m \begin{pmatrix} \dfrac{1}{m} + 2y_1 \dot{y}_2 & 2y_2 \dot{y}_2 \\ -2y_1 \dot{y}_1 & -2y_2 \dot{y}_1 + \dfrac{1}{m} \end{pmatrix} \right]. \tag{5}$$

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The $\boldsymbol{Y}$-independent part of F is obviously symmetric if the scalar product is taken as a combination of matrix scalar product and time integration with zero values for the perturbations u and v at the integration limits. The other part reduces to testing the expression

$$\begin{aligned}&\int \mathrm{d}t\, u' \{2n\, \dot y_1\, \dot y_2\, \dot u + [m(y_1^2+y_2^2) + n(\dot y_1^2 + 3\dot y_2^2)]\, \dot v\}\\&+\int \mathrm{d}t\, v'\{-[m(y_1^2+y_2^2)+n(3\dot y_1^2+\dot y_2^2)]\,\dot u - 2n\,\dot y_1\,\dot y_2\,\dot v\}\\&+\int \mathrm{d}t\, m[u'(2y_1\,\dot y_2\, u + 2y_2\,\dot y_2\, v)\\&\qquad - v'(2y_1\,\dot y_1\, u + 2y_2\,\dot y_1\, v)]\,. \qquad (6)\end{aligned}$$

F is symmetric if (6) is invariant with respect to the interchange of the couples (u', v') and (u, v). This is obviously not the case if $n \neq 0$ because the invariance is violated, particularly with respect to interchanging of $(u', 0)$ and $(u, 0)$. For $n = 0$ after integrating by parts the first m term, (6) becomes

$$\begin{aligned}&-\int \mathrm{d}t\, m(y_1^2+y_2^2)\, \dot u'\, v + v'\, \dot u)\\&+\int \mathrm{d}t\,[2m\,(y_1\,\dot y_2\, u'\, u - y_2\,\dot y_1\, v'\, v)]\\&-\int \mathrm{d}t\, 2m\, y_1\,\dot y_1\,(u'\, v + u\, v')\,, \qquad (7)\end{aligned}$$

which is obviously symmetric.

This means that in terms of y_1 and y_2 there is a Lagrangean for system (3, 4) only if $n = 0$. It is given by

$$\begin{aligned}\mathscr{L} &= \int L\, \mathrm{d}t\\&= \int\left[\frac{1}{2}(\dot y_1^2+\dot y_2^2) - \frac{1}{2}(y_1^2+y_2^2) + \frac{m}{3}(y_1^3+y_2^3)(\dot y_1 - \dot y_2)\right.\\&\qquad\left. + \frac{p}{2}(y_1+y_2)(\dot y_1 - \dot y_2)\right]\mathrm{d}t\,. \qquad (8)\end{aligned}$$

The canonically conjugate momenta to y_1 and y_2 are

$$\begin{aligned}p_1 &= \frac{\partial L}{\partial \dot y_1} = \dot y_1 + \frac{1}{3}m(y_1^3+y_2^3) + \frac{p}{2}(y_1+y_2)\,,\\p_2 &= \frac{\partial L}{\partial \dot y_2} = \dot y_2 - \frac{1}{3}m(y_1^3+y_2^3) - \frac{p}{2}(y_1+y_2)\,.\end{aligned} \qquad (9)$$

The Hamiltonian can be obtained from

$$H = \sum_{i=1}^{2} p_i\, \dot y_i - L\,.$$

This is a rather long expression if written in the p_i and y_i but becomes very simple if written noncanonically in terms of $\dot y_i$ and y_i. It is then

$$H = \tfrac{1}{2}(\dot y_1^2+\dot y_2^2) + \tfrac{1}{2}(y_1^2+y_2^2)\,. \qquad (10)$$

Note that a quadratic expression of the type (10) is a constant of motion in a very general way even for system (1). This can easily be seen by forming the scalar product of system (1) with $\dot{\boldsymbol{Y}}$, the contributions of M_a, N_a, and P_a being zero because of their antisymmetry.

This suggests that a noncanonical description be sought for (3) and (4). Writing them as first-order equations in terms of x_i where

$$x_1 = y_1\,,\quad x_2 = y_2\,,\quad x_3 = \dot y_1\,,\quad x_4 = \dot y_2\,, \qquad (11)$$

we obtain

$$\begin{pmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & -\Delta\\ 0 & -1 & \Delta & 0\end{pmatrix}\begin{pmatrix}\dfrac{\partial H}{\partial x_1}\\ \vdots\\ \dfrac{\partial H}{\partial x_4}\end{pmatrix} \qquad (12)$$

with

$$H = \frac{1}{2}\sum_{i=1}^{4} x_i^2 \qquad (13)$$

and

$$\Delta = m(x_1^2+x_2^2) + n(x_3^2+x_4^2) + p\,. \qquad (14)$$

Equation (12) can also be written symbolically as

$$\dot x_i = A_{ij}\frac{\partial H}{\partial x_j} \quad \text{or} \quad \dot{\boldsymbol{X}} = \mathbf{A}\frac{\partial H}{\partial \boldsymbol{X}}\,. \qquad (15)$$

It can easily be seen that $\det(A) = 1$ and

$$A^{-1} = \begin{pmatrix}0 & -\Delta & -1 & 0\\ \Delta & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}. \qquad (16)$$

Equation (15) is a Hamiltonian system only if the A_{ij} can build up a Poisson structure

$$[f, g] = \frac{\partial f}{\partial x_i} A_{ij}\frac{\partial g}{\partial x_j}\,. \qquad (17)$$

In this case the A_{ij} are quadratic in the dynamic variables and the bracket (17) is not of the Lie-Poisson type. So we have to check Jacobi's identity, which reduces to the condition [5]

$$\frac{\partial A_{ij}^{-1}}{\partial x_k} + \frac{\partial A_{jk}^{-1}}{\partial x_i} + \frac{\partial A_{ki}^{-1}}{\partial x_i} = 0 \qquad (18)$$

for all i, j, k. In our case it reduces to

$$\begin{aligned} -\frac{\partial \Delta}{\partial x_1}+\frac{\partial \Delta}{\partial x_1}+0=0\,, \\ -\frac{\partial \Delta}{\partial x_2}+0+\frac{\partial \Delta}{\partial x_2}=0\,, \\ -\frac{\partial \Delta}{\partial x_3}+0+0=0\,, \\ -\frac{\partial \Delta}{\partial x_4}+0+0=0\,. \end{aligned} \tag{19}$$

The first two conditions are identically verified but the last two conditions require $n=0$.

We find again that $n=0$ is needed even for a noncanonical Hamiltonian formulation. This is proved for (13) taken as Hamiltonian. In fact system (3, 4) is completely integrable [2] even for $n\neq 0$ and owing to theorems of dynamics [6] it should have a Hamiltonian formulation. But the Hamiltonian introduced in that proof contains all constants of motion and possibly variables other than the x_i. The situation for $r=2$ is now clear. For $n=0$ we have rather simple canonical and noncanonical formulations for system (3, 4). For $n\neq 0$ we have to invoke complete integrability [2] to prove the existence [6] of a Hamiltonian, but we know in advance that it is going to be cumbersome.

III. Case $r>2$

For $r>2$ and $n\neq 0$ we cannot expect to apply the previous theorems of dynamics of [6] because we know from [3] that system (1) is in general nonintegrable. However, integrability is sufficient but not necessary to have a Hamiltonian formulation. The easiest approach is therefore to try a noncanonical formalism for $N_a=0$ with a quadratic Hamiltonian of the type (13). As in the case $r=2$, we introduce

$$x_1=y_1,\dots,x_r=y_r;\quad x_{r+1}=\dot y_1,\dots,x_{2r}=\dot y_r\,. \tag{20}$$

System (1) becomes

$$\begin{pmatrix} \dot x_1 \\ \vdots \\ \vdots \\ \dot x_r \\ \dot x_{r+1} \\ \vdots \\ \vdots \\ \dot x_{2r} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & & \vdots & 0 & & & & \vdots \\ \vdots & & & \vdots & \vdots & & & & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 & -\Delta_{1,2} & \cdots & & -\Delta_{1,r} \\ 0 & & & \vdots & \Delta_{1,2} & & & & \vdots \\ \vdots & & & 0 & \vdots & & & & -\Delta_{r-1,r} \\ 0 & \cdots & 0 & -1 & \Delta_{1,r} & \cdots & & \Delta_{r-1,r} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \frac{\partial H}{\partial x_{2r}} \end{pmatrix} \tag{21}$$

with

$$H=\frac{1}{2}\sum_{i=1}^{2r} x_i^2 \tag{22}$$

and

$$\Delta_{i,j}=m_{ij}\sum_{k=1}^{r} x_k^2-p_{ij}\,, \tag{23}$$

where m_{ij} and p_{ij} are the elements of M_a and P_a, respectively, N_a being taken equal to zero. System (21) can be written symbolically as

$$\dot x_i = A_{ij}\frac{\partial H}{\partial x_j}\quad \text{or}\quad \dot{\boldsymbol X}=\mathbf{A}\frac{\partial H}{\partial \boldsymbol X}\,. \tag{24}$$

It is easy to see that $\det(A)=1$ and $A^{-1}=$

$$\begin{pmatrix} 0 & -\Delta_{1,2} & \cdots & & -\Delta_{1,r} & -1 & 0 & \cdots & 0 \\ \Delta_{1,2} & & & & \vdots & 0 & & & \vdots \\ \vdots & & & & -\Delta_{r-1,r} & \vdots & & & 0 \\ \Delta_{1,r} & \cdots & & \Delta_{r-1,r} & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & & 0 & 0 & \cdots & \cdots & 0 \\ 0 & & & & \vdots & \vdots & & & \vdots \\ \vdots & & & & 0 & \vdots & & & \vdots \\ 0 & \cdots & & 0 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{25}$$

We now prove that conditions (18) cannot be verified by checking, for example,

$$\frac{\partial A_{12}^{-1}}{\partial x_3}+\frac{\partial A_{23}^{-1}}{\partial x_1}+\frac{\partial A_{31}^{-1}}{\partial x_2}=0\,. \tag{26}$$

The three terms are proportional to x_3, x_1 and x_2, respectively. They cannot cancel each other.

A Langrangean formulation in terms of $\boldsymbol{Y}$ is not possible either. One arrives at that conclusion after calculating the Frechet derivative of system (1). It is very similar to the calculation done in Sect. II, but it is lengthier. It turns out that the Frechet derivative cannot be symmetric unless all nonlinear terms of system (1) are identically zero. This is more restrictive than in the case $r=2$ where only the n term had to be zero. Note that for $N_a \neq 0$ the previous calculations would be even more complicated and the answer would be negative, as can already be seen from the case $r=2$.

IV. Statistics and Final Remarks

System (1) is the attractor of a driven, damped Van der Pol type system introduced in [1]. The latter system, if linearized, would give unstable eigenvalues for all oscillators. In this respect it cannot model a successive onset of Hopf bifurcations [7], for which only one oscillator at a time becomes unstable by changing a bifurcation parameter. But it can model the situation where all or most of the eigenmodes are linearly unstable and saturate at a level which is given by system (1). In this respect the situation is similar to turbulence with large Reynolds numbers and not to a gradual increase in disorder. The statistics of system (1) is somehow similar to fully developed turbulence, but is far easier to do.

Conventional equilibrium statistics would require a canonical Hamiltonian for system (1). A noncanonical Hamiltonian would also be sufficient, as noted in [8]. But in the case of system (1) we were not able to find any Hamiltonian at all for $r>2$. The case $r=2$, which has a Hamiltonian, is obviously unsuited to do statistics.

A way out of the situation is first to look for a Liouville theorem independently of Hamiltonian formulation and secondly to have some positive definite constant of motion. The positive constant of motion has already been mentioned, and is given by (22). The Liouville theorem reduces to proving incompressibility in phase space. This also is easily seen from the definition (20) and system (1),

$$\sum_{i=1}^{2r}\frac{\partial \dot{x}_i}{\partial x_i}=\sum_{i,j=r+1}^{2r} 2\,x_i\,n_{ij}\,x_j=0\,, \tag{27}$$

where n_{ij} are the elements of the antisymmetric matrix N_a.

This together with an ergodicity assumption allows us to introduce a microcanonical distribution centered at a particular value of (22). The passage from a microcanonical to a canonical distribution requires the exchange of fluctuations of H (given by (22) and is not the Hamiltonian) with a "heat bath" in such a way that the average value of H is a given constant. A remarkable result is that despite strong nonlinearities in system (1) there is equipartition among the oscillators amplitudes due to the quadratic form of (22).

Let us finally note that if system (1) happens to have a Hamiltonian formulation in variables other than the x_i, it is to be expected that it will be very cumbersome (we know this for the case $r=2$) and it will not readily lead to a simple result concerning the statistics.

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